Higher-order initial-value problems

1. Convert this 2nd-order IVP into a system of two first-order IVPs

$$y^{(2)}(t) = \sin(t) - 2y^{(1)}(t) - 3y(t)$$
$$y(0) = 7$$
$$y^{(1)}(0) = -5$$

$$\mathbf{w}(t) = \begin{pmatrix} w_0(t) \\ w_1(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ y^{(1)}(t) \end{pmatrix}$$

Answer: $\mathbf{w}^{(1)}(t) = \begin{pmatrix} w_1(t) \\ \sin(t) - 2w_1(t) - 3w_0(t) \end{pmatrix}$

$$\mathbf{w}(0) = \begin{pmatrix} 7 \\ -5 \end{pmatrix}$$

2. Convert this 3rd-order IVP into a system of three first-order IVPs

$$u^{(3)}(t) = -tu^{(2)}(t)u(t)$$
$$u(0) = 7$$
$$u^{(1)}(0) = 4$$
$$u^{(2)}(0) = 2$$

$$\mathbf{w}(t) = \begin{pmatrix} w_0(t) \\ w_1(t) \\ w_2(t) \end{pmatrix} = \begin{pmatrix} u(t) \\ u^{(1)}(t) \\ u^{(2)}(t) \end{pmatrix}$$

Answer: $\mathbf{w}^{(1)}(t) = \begin{pmatrix} w_1(t) \\ w_2(t) \\ -tw_2(t)w_0(t) \end{pmatrix}$

$$\mathbf{w}(0) = \begin{pmatrix} 7 \\ 4 \\ 2 \end{pmatrix}$$

3. Convert this 4th-order IVP into a system of four first-order IVPs

$$x^{(4)}(t) = -5\left(x^{(2)}(t)+1\right)\left(x^{(3)}(t)+x(t)+t\right)$$
$$x(0) = 2$$
$$x^{(1)}(0) = -4$$
$$x^{(2)}(0) = 5$$
$$x^{(3)}(0) = 3$$

$$\mathbf{w}(t) = \begin{pmatrix} w_0(t) \\ w_1(t) \\ w_2(t) \\ w_3(t) \end{pmatrix} = \begin{pmatrix} x(t) \\ x^{(1)}(t) \\ x^{(2)}(t) \\ x^{(3)}(t) \end{pmatrix}$$

Answer: $$\mathbf{w}^{(1)}(t) = \begin{pmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \\ -5\left(w_2(t)+1\right)\left(w_3(t)+w_0(t)+t\right) \end{pmatrix}$$

$$\mathbf{w}(0) = \begin{pmatrix} 2 \\ -4 \\ 5 \\ 3 \end{pmatrix}$$

4. In all the above questions, we always wrote that the nth derivative is described in terms of lower derivatives and the function itself. Is this a fair assumption to make? Is it not possible that we may have a system where this is not the case?

Answer: Yes, but the implicit function theorem says that this is possible essentially everywhere.

5. How would you express the right-hand side of the ode to a function in Matlab?

Answer:

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>> f = @(t, w)( [w(2); w(3); w(4); -5*(w(3) + 1)*(w(4) + w(1) + t)] );
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You must remember that in Matlab, vectors start at index 1, but for higher-order odes, it is often easier to understand if you represent the kth derivative of the function by $w_k(t)$.

6. For Question 4, if you were to approximate $\mathbf{w}(t_k)$ with $\mathbf{w}_k$ where $t_k = t_0 + kh$, at each step, you would have a 4-dimensional vector. What do the entries of that 4-dimensional vector represent?

Answer: The first entry of the approximation $\mathbf{w}_k$ would approximate $x(t_k)$.

The second entry of the approximation $\mathbf{w}_k$ would approximate $x^{(1)}(t_k)$.

The third entry of the approximation $\mathbf{w}_k$ would approximate $x^{(2)}(t_k)$.

The fourth entry of the approximation $\mathbf{w}_k$ would approximate $x^{(3)}(t_k)$.

7. Suppose you had approximations to the solution as described in Question 6. Suppose you also wanted to approximate the solution on the interval $[t_k, t_{k+1}]$. Could you find a polynomial that matches the approximations of the function at t_k and t_{k+1} and also matches all four derivatives at each of the points?

Answer: In theory, yes, but you would have to use a spline of degree 9.